Chapter 7
Techniques of Integration

The following notes are for the Calculus C (SDSU Math 151) classes I teach at Torrey Pines High School. I wrote and modified these notes over several semesters. The explanations are my own; however, I borrowed several examples and diagrams from the textbooks my classes used while I taught the course. Over time, I have changed some examples and have forgotten which ones came from which sources. Also, I have chosen to keep the notes in my own handwriting rather than type to maintain their informality and to avoid the tedious task of typing so many formulas, equations, and diagrams. These notes are free for use by my current and former students. If other calculus students and teachers find these notes useful, I would be happy to know that my work was helpful. - Abby Brown*

**Calculus: Early Transcendentals*, 6th & 4th editions, James Stewart, ©2007 & 1999 Brooks/Cole Publishing Company, ISBN 0-495-01166-5 & 0-534-36298-2.
(Chapter, section, page, and formula numbers refer to the 6th edition of this text.)

**Calculus*, 5th edition, Roland E. Larson, Robert P. Hostetler, & Bruce H. Edwards, ©1994 D. C. Heath and Company, ISBN 0-669-35335-3.

SDUHSD Calculus II/C
SDSU Math 151

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Name: ______________________________

INTEGRATION BY PARTS (page 1 of 2)

Ex: $\int x e^{2x} dx$ $\longrightarrow$ Try u-sub. $u =$
$du =$

Recall: u-substitution lets us reduce an integral to solve it with basic methods.

→ u-substitution is to integration as ________________ is to differentiation.

We need another approach.

→ Integration by parts is to integration as ________________ is to differentiation.

Consider the product of two functions $u = u(x)$ and $v = v(x)$.

$$\frac{d}{dx}[uv] = \frac{du}{dx} v + u \frac{dv}{dx}$$

↓ Integrate both sides of the product rule

$$uv = \int \qquad dx + \int \qquad dx$$

$$uv =$$

$$\boxed{\int u \, dv = \qquad\qquad}$$ ← Integration by Parts

Solving by parts feels like more complicated u-substitution.

Ex: $\int x e^{2x} dx$ $\quad u = \quad du = \quad dv = \quad v =$

$\therefore \int x e^{2x} dx =$

Choosing u and dv becomes easier with practice.

Guidelines

- Try dv as the complicated part that fits basic integration.
- Try u as the part with the simpler derivative.

Integration by Parts – page 2

Name: ______________________

Ex: $\int \ln x \, dx$

$u =$ $\qquad dv =$

$du =$ $\qquad v =$

$\therefore \int \ln x \, dx =$

Try: $\int x^3 \ln x \, dx$

Sometimes you will need parts more than once.
(along with a clever observation)

Ex: $\int e^{3x} \cos 2x \, dx$

Try: ① $\int x \arctan x \, dx$

② $\int_0^1 x\, 5^x \, dx$

Tips: - For $\int x^n e^{ax} dx$, $\int x^n \sin ax \, dx \rightarrow$ Let $u = x^n$, $dv =$ rest

- For $\int x^n \ln x \, dx$, $\int x^n \arcsin ax \, dx \rightarrow$ Let $u = \ln x$ or $\arcsin ax$

Trigonometric Integrals (page 1 & 2)

Formulas: $\cos^2 x + \sin^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$
$\cos^2 x = 1 - \sin^2 x$

Along with $\cos^2 x - \sin^2 x = \cos 2x$ we get

☆ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
☆ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

} Half-Angle Identities

Also, $\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} \Rightarrow 1 +$

or ☆ $\tan^2 x =$

and ☆ $\sec^2 x =$

Don't forget: $\int \sin x\,dx = -\cos x + C$ and $\int \sec^2 x\,dx = \tan x + C$
$\int \cos x\,dx = \sin x + C$ $\quad \int \sec x \tan x\,dx = \sec x + C$
etc.

We will be solving integrals in the forms:
$\int \sin^m x \cos^n x\,dx$ and $\int \sec^m x \tan^n x\,dx$

Ex: $\int \sec^4 x \tan x\,dx$

$u =$
$du =$

We need to "save" part of the integrand for the du.

Sometimes we will need to convert one trig. function into another.

Ex: $\int \cos^3 \frac{x}{3}\,dx$

Think: If we let __________ $= u$
then we can use ____________ for du.

Check answer with derivative.

Strategies for $\int \sin^m x \cos^n x \, dx$

If cosine power is odd, "save" one and convert rest to sine. $u = \sin x$

If sine power is odd, "save" one and convert rest to cosine. $u = \cos x$

If both are even, use half-angle identities or $\sin x \cos x = \frac{1}{2}\sin 2x$.

Ex: $\int \cos^3 x \sin^2 x \, dx$

Ex: $\int \cos^2 x \sin^2 x \, dx$

Strategies for $\int \tan^m x \sec^n x \, dx$

If secant power is even, "save" two and convert rest to tangent. $u = \tan x$

If tangent power is odd, "save" $\sec x \tan x \, dx$ and convert rest to secant. $u = \sec x$

Ex: $\int \tan^3 x \sec^3 x \, dx$

Also note:

- Cotangent and cosecant form a pair similar to tangent and secant. Watch out for negative signs!
- To form sin–cos, tan–sec, and cot–csc pairs, you may need $\tan x = \frac{\sin x}{\cos x}$, $\csc x = \frac{1}{\sin x}$, etc.
- You can check your answers with derivatives.
- There are additional examples in the book.

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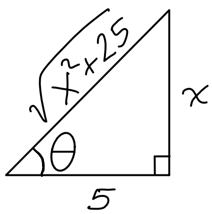
TRIGONOMETRIC SUBSTITUTION (page 1 of 2)

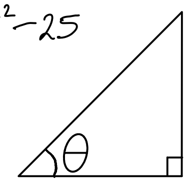
Integrals such as $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ cannot be solved with u-substitution either. We will use triangles to convert the integral to trig functions of θ, integrate, then switch back to x.

Given a square root in the form of $\sqrt{a^2 \pm u^2}$, we can use the Pythagorean Theorem to create a substitution triangle.

Given: $\sqrt{x^2+25}$

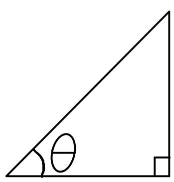


Given: $\sqrt{x^2-25}$



Hint: If x is the hypotenuse, place root opposite θ. If x is not hyp., place x opp. θ. This avoids cot & csc.

Ex: $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ First, create triangle.
Second, use SohCahToa to set up substitution.



$\frac{x}{5} =$

$x =$

$dx =$

$\sqrt{25-x^2} =$

Use the side with the constant to convert x and root to θ.

$\therefore \int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int$

$=$

Don't forget to switch θ back to x after integrating. For definite integrals, you can switch the limits to θ to avoid the last substitution.

Ex: $\int \frac{\sqrt{4x^2+9}}{x^4}\,dx$

Trigonometric Substitution – page 2

Ex: $\int \frac{x^2}{(1+x^2)^2}\,dx$ Where's the root?

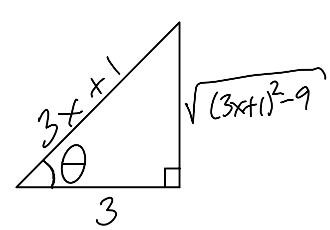
Try: ① $\int \frac{\sqrt{x^2-4}}{x}\,dx$

② $\int \frac{1}{x\sqrt{16-4x^2}}\,dx$

Sometimes you need to complete the square.

Ex: $\sqrt{9x^2+6x-8}$

$= \sqrt{9x^2+6x\mathbf{+1}-8\mathbf{-1}} = \sqrt{(3x+1)^2-9}$



PARTIAL FRACTIONS (page 1 of 2)

$\int \frac{1}{x^2-5x+6}\,dx$ ← To solve 1. Complete square
2. Trigonometric Substitution
Long and complicated...

OR Notice that:

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}$$

So $\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{x-3} - \frac{1}{x-2}\,dx =$

But how do we figure out how to split it up? It's more than just factoring.

Method of Partial Fractions $\frac{N(x)}{D(x)}$

1. If degree of N ≥ degree of D, divide then apply the following.
2. Factor denominator.
   Linear Factors: $(px+q)^m$, Quadratic Factors: $(ax^2+bx+c)^n$
3. For each linear factor $(px+q)^m$ decompose
   $$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \cdots + \frac{A_m}{(px+q)^m}$$
4. Quadratic Factors $(ax^2+bx+c)^n$ decompose
   $$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

DON'T WORRY. IT'S NOT AS COMPLICATED AS THIS SEEMS.

Back to example: $\frac{1}{x^2-5x+6} \Rightarrow x^2-5x+6 = (\qquad)(\qquad)$

$$\therefore \frac{1}{x^2-5x+6} = \frac{A}{\qquad} + \frac{B}{\qquad}$$

Multiply both sides by common denominator.
⇒ Basic equation :
plug in easy numbers to solve for A and B.

Partial Fractions – page 2

Ex: $\int \frac{2x-3}{x^2-2x+1}\,dx$

★ For quadratic factors, it is often easier to collect and compare like terms.

Ex: $\int \frac{x^2-4x+7}{x^3-x^2+x+3}\,dx$ Hint: $x^3-x^2+x+3 = (x+1)(x^2-2x+3)$

Try: $\int \frac{2x^2+x+8}{(x^2+4)^2}\,dx$

Note: Treat x^2 as a linear factor: $\frac{A}{x}+\frac{B}{x^2}$.
Also, factor using integers. For example, do NOT use $(x^2-3) = (x-\sqrt{3})(x+\sqrt{3})$

INTEGRATION BY TABLES (page 1 of 2)

As you may have already noticed, the back pages of the textbook contain 120 formulas for integration. Using tables isn't always as easy as it may seem. You will often need to use u-substitution or a second or third formula.

Table Categories: Basic forms, $\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$, $(a+bu)$, Trig. Fcns., Inverse Trig. Fcns., Exponents and Logarithms, Hyperbolic Fcns., and $\sqrt{2au-u^2}$.

→ "a" is constant, "u" is function.

Ex: $\int \frac{x}{\sqrt{1+x}}\,dx$ Let's try "a+bu".

Formula 55: $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

For our example: $u=$ $a=$

$du=$ $b=$

$\therefore \int \frac{x}{\sqrt{1+x}}\,dx =$

Ex: $\int x^2\sqrt{2+9x^2}\,dx$ Once you find a formula, rewrite integral to match.

Formula ____:

Ex: $\int x^3 \sin x\,dx$ ← We can do this one by parts.

Integration by Tables – page 2

$\int x^3 \sin x\,dx$ $\quad u = x^3 \quad dv = \sin x\,dx$
$\quad du = 3x^2\,dx \quad v = -\cos x$

Integration by Parts THREE times.

$= -x^3\cos x + \int 3x^2 \cos x\,dx$ $\quad u = x^2 \quad dv = \cos x\,dx$
$\quad du = 2x\,dx \quad v = \sin x$

$= -x^3\cos x + 3(x^2 \sin x - \int 2x \sin x\,dx)$ $\quad u = x \quad dv = \sin x\,dx$
$\quad du = dx \quad v = -\cos x$

$= -x^3\cos x + 3x^2\sin x - 6(-x\cos x + \int \cos x\,dx)$

$= -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$

$= (6x - x^3)\cos x + (3x^2 - 6)\sin x + C$

Reduction Formulas can simplify this.

Reduction formulas include a new integral with a smaller degree.
(Don't forget to account for $dx \to du$ conversion, if needed.)

Formula 82: $\int u \sin u\,du = \sin u - u\cos u + C$
Formula 83: $\int u \cos u\,du = \cos u + u \sin u + C$
Formula 84: $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1} \cos u\,du$
Formula 85: $\int u^n \cos u\,du = u^n \sin u - n \int u^{n-1} \sin u\,du$

Examples of Reduction Formulas

Ex: $\int x^3 \sin x\,dx$

On a test or a quiz, I may give you an integral and a short list of formulas. This would test if you can apply the formulas correctly.

Note: Understanding how to apply the formulas is an important skill. However, throughout the year you should do all integrals by hand. Refer to the tables only when stuck.

L'Hôpital's Rule (page 1 of 2)

Indeterminant Forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^{∞}, ∞^0, 0^0, $\infty - \infty$

As limits they do not guarantee that a limit exists, nor do they indicate what the limit is.

☆ Algebraically: $\lim\limits_{x \to 3} \frac{x-3}{x^2-9} =$

What about $\lim\limits_{x \to 0} \frac{e^{2x}-1}{x}$?

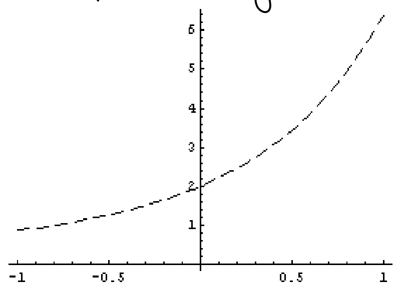
Direct substitution gives $\frac{e^{2(0)}-1}{0} \longrightarrow$

$\underline{OR}$ $\lim\limits_{x \to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x} \right) = \lim\limits_{x \to 0} \frac{e^{2x}}{x} - \lim\limits_{x \to 0} \frac{1}{x} \longrightarrow$

☆ Numerically:

x	-1	-0.01	-0.001	0	0.001	0.01	1
$\frac{e^{2x}-1}{x}$	0.865	1.980	1.998	Undef.	2.002	2.020	6.389

☆ Graphically:



So what does all of this tell us about the limit? What is the difference between the "limit" and the "value" of the function at $x=0$? How can we find the limit algebraically?

For the indeterminate forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ apply
$\underline{\text{L'Hôpital's Rule}}$ $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$

☆ ONLY FOR $\frac{0}{0}$ and $\frac{\infty}{\infty}$ FORMS ☆

Ex: $\lim\limits_{x \to 0} \frac{e^{2x}-1}{x} = \lim\limits_{x \to 0} \_\_\_\_\_\_ =$

This also works when $x \to \pm\infty$.

Ex: $\lim\limits_{x \to \infty} \frac{3x^2-1}{2x^2+1} =$

L'Hôpital's Rule – page 2

L'Hôpital's Rule is proved using the

<u>Extended Mean Value Theorem</u>: If f and g are differentiable on (a,b) and continuous on $[a,b]$ such that $g'(x) \neq 0$ then there is a point c in (a,b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note: If $g(x) = x$, you have the MVT.

Roughly, $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{\Delta f}{\Delta g} = \dfrac{\Delta f}{\Delta g} \cdot \dfrac{1/\Delta x}{1/\Delta x} = \dfrac{\frac{\Delta f}{\Delta x}}{\frac{\Delta g}{\Delta x}} \xrightarrow{\text{with limits}} \dfrac{\frac{df}{dx}}{\frac{dg}{dx}}$

There is some c in (a,b) where the ratio of the changes in f and g is equal to the ratio of the slopes at c.

Ex: ① $\displaystyle\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ ② $\displaystyle\lim_{x\to\infty} e^{-x}\sqrt{x}$

Use logarithms to solve limits with variable exponents.

③ $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ ④ $\displaystyle\lim_{x\to 0} (2x+1)^{1/x}$

⑤ $\displaystyle\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$ ⑥ $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$ ⑦ $\displaystyle\lim_{x\to 0} \frac{e^x}{x}$

Improper Integrals (page 1 of 2)

The FTC requires $f(x)$ to be continuous on $[a,b]$ for $\int_a^b f(x)\,dx$ to be defined.
There are two types of Improper Integrals:

(A) $\int_a^\infty f(x)\,dx$ — Infinite limit or limits

(B) $\int_a^b f(x)\,dx$ — $f(x)$ not continuous somewhere in $[a,b]$

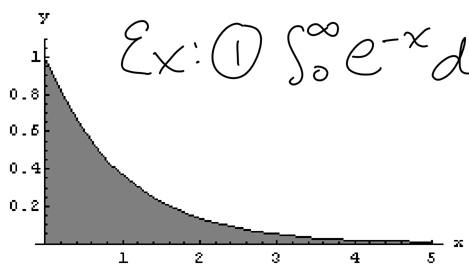
So what do we do?
We cannot just plug ∞ into the results. We must use limits.

(A) $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ solve integral, then take limit.

(B) $\int_a^b f(x)\,dx$ with f not continuous at $c \in (a,b)$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{s\to c^+} \int_s^b f(x)\,dx$$

Break apart,
Convert to limits,
solve integrals, find limits.



Ex: (1) $\int_0^\infty e^{-x}\,dx =$

When the limit exists, we say that the integral converges.
When the limit does not exist, we say the integral diverges. (This includes integrals whose limit is $\pm\infty$.)

(2) $\int_0^\infty \sin x\,dx =$

(3) $\int_{-1}^{2} \frac{1}{x^3}\,dx =$

(4) $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$ (set up):
To integrate, let $u=\sqrt{x}$.

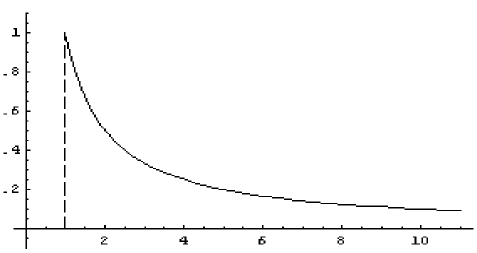
(5) $\int_1^\infty (1-x)e^{-x}\,dx = \left(\lim_{b\to\infty} \frac{b}{e^b}\right) - \frac{1}{e}$ This includes integration by parts. How do we solve the limit?

(6) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ ← Choose a convenient value in interval for splitting.

Special Improper Integral: $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$

Easily proved using the power rule.

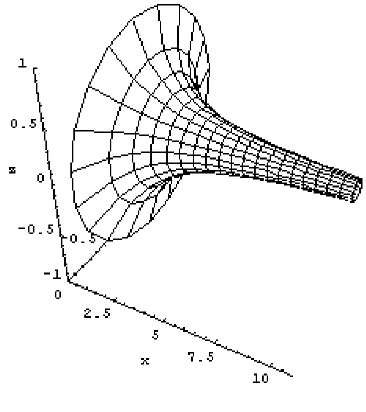
Example: "Gabriel's Horn" Region $y=\frac{1}{x}$, $y=0$, $x \ge 1$ revolved about the x-axis.



Surface Area $= 2\pi \int_1^\infty f(x)\sqrt{1+(f'(x))^2}\,dx$

This integral diverges.

∴ Surface area is infinite.



However, using the disc method, we can find

Volume $= \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$

$= \pi\ ($ $\qquad\qquad\qquad$ $)$ using formula

$=$

∴ The volume is ______________!

Chapter 8
Further Applications of Integration

8-1 Arc Length
8-2 Area of a Surface of Revolution

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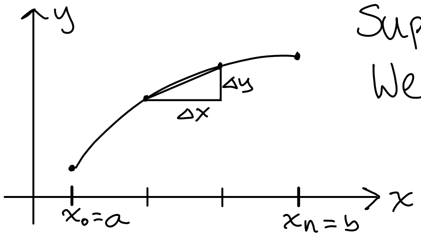
ARC LENGTH (page 1 of 1)



Suppose we want to find the length of a curve. We can break it into pieces and approximate each piece by a line segment.

Endpoints of one segment: $(x_i, y_i), (x_{i+1}, y_{i+1})$

Let $\Delta x_i = x_{i+1} - x_i$ and $\Delta y = y_{i+1} - y_i$

Length of segment: Distance formula $= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

Length of entire curve is approximated by the sum $\sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

With a little algebra, this becomes $\sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i)$

But to get the exact length we need the limit as $\Delta x \to 0$ or $n \to \infty$.

Note: $\lim_{n \to \infty} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$

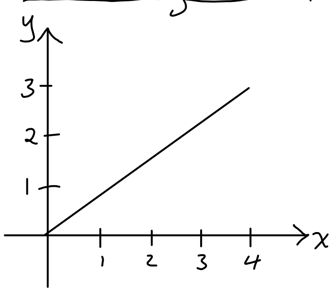
lower-case

Arc Length $= s =$ for $y = f(x)$ on $[a, b]$
and $s =$ for $x = g(y)$ on $[c, d]$.

Note that this is just a general sketch of the proof which depends on the Mean Value Theorem.

Let's check this formula with familiar curves.

Line Segment from $(0,0)$ to $(4,3)$:



Pythagorean Theorem $\Rightarrow$ Arc Length $= s =$

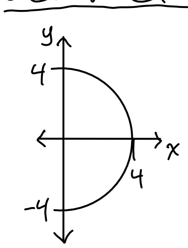
Equation of line: $f(x) =$

$f'(x) =$

Arc length formula $\Rightarrow s =$

$=$

Semi-Circle: $\frac{1}{2}$ Circumference $= \frac{1}{2} \cdot 2\pi r =$



$x^2 + y^2 = 16$

$x = \sqrt{16 - y^2} \rightarrow g(y) = (16 - y^2)^{1/2}$

$g'(y) =$

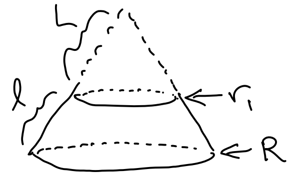
$s =$

Arc Length Function: $s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$

$\therefore \frac{ds}{dx} = \sqrt{1 + (f'(x))^2} \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\, dx \Rightarrow s = \int ds$

Area of a Surface of Revolution (page 1 of 2)

First we need to find the lateral surface area of the frustum of a cone.



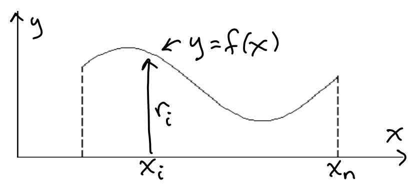
Lateral surface area of large cone; Lat. surf. area of top cone

$$\text{Surface Area of Frustum} = \pi R(l+L) - \pi r_1 L = \pi((R-r_1)L + Rl)$$

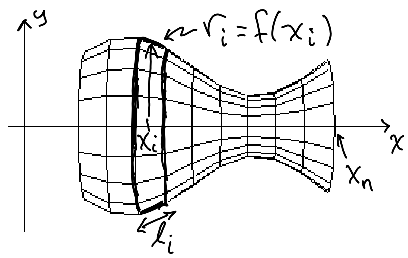
similar triangles $\Rightarrow \frac{L}{r_1} = \frac{l+L}{R} \Rightarrow LR = r_1 l + r_1 L \Rightarrow L(R-r_1) = r_1 l$

$\therefore$ Surf. Area of Frustum $= \pi(r_1 l + Rl)$. Let $r = \text{avg. radius} = \frac{r_1+R}{2}$

$$\boxed{\text{Surf. Area of Frustum} = 2\pi r l}$$



Given a surface of revolution, imagine breaking it into strips. Each strip can be approximated as a narrow frustum. (about the x-axis)



The lateral surface area of strip $i = 2\pi r_i l_i$

$l_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i)$ using approx. for arc length.

$\therefore$ Area approximation $= \sum_{i=1}^{n} 2\pi r_i l_i = \sum_{i=1}^{n} 2\pi f(x_i)\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$

Again, we need to take the limit as $\Delta x_i \to 0$ or $n \to \infty$ for <u>exact</u> answer.

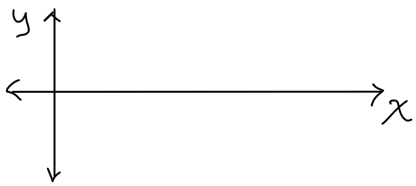
Area of Surface of Revolution about the x-axis $= S =$ (upper-case)   for $y = f(x)$ on $[a,b]$.

Note that this is a general sketch and not a formal proof.

Keeping $A = 2\pi r l$ in mind, we can develop four cases.

	curve: $y=f(x)$ on $[a,b]$ use dx	curve: $x=f(y)$ on $[c,d]$ use dy
Rotation about x-axis	$r=f(x)$ $S=$	$r=y$ $S=$
Rotation about y-axis	$r=x$ $S=$	$r=f(y)$ $S=$

Ex: "Gabriel's Horn" Region $y=\frac{1}{x}$, $y=0$, $x\geq 1$
Revolved about the x-axis.



$$S=2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$
$$=2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$$

Since $\sqrt{x^4+1}>\sqrt{x^4}$,

$$S>2\pi\int_1^\infty \frac{x^2}{x^3}\,dx=$$

Ex: $x=1+2y^2$, $1\leq y\leq 2$ rotated about the x-axis.

Calculus C – Exam #1 Review

Chapter 7 – Techniques of Integration

Larson 7.1 Review of Basic Techniques

7.1 Integration by Parts

7.2 Trigonometric Integrals

7.3 Trigonometric Substitution

7.4 Integration of Rational Functions by Partial Fractions

7.5 Strategy for Integration

7.6 Integration Using Tables

Larson 7.7 L'Hôpital's Rule (Remember: Only 0/0 or ±∞/±∞ forms)

7.8 Improper Integrals (Don't Forget: Write $\lim\limits_{b\to\infty}$, etc. where needed.)

Chapter 8 – Further Applications of Integration

8.1 Arc Length

8.2 Area of a Surface of Revolution

Integration: How am I supposed to know which method to use?

0th **Multiply out powers to get several small integrals**

1st **Look for a possible *u*-substitution**

Pick some "inside" function

Is the derivative of your u available to become part of du?

2nd **Can the integrand be adjusted slightly so that a basic *u*-substitution will work?**

Multiply by a constant of by "1" with variables

Adding "0" by (+) and (-) a constant

3rd **Can the integral be split?**

Simple numerator split

Partial Fractions Decomposition

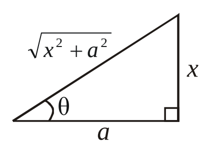
4th **Is the integrand an improper rational expression?**

Use long division to split

5th **Is it inverse trig.? Or Trigonometric Substitution?**

arctan, arcsin, arcsec

Draw a triangle (3 types)



6th **Can I complete the square to make it look like inverse trig.?**

Don't forget to also subtract anything new you add

7th **Would trig. identities help?**

$\cos^2 x + \sin^2 x = 1$ $\quad$ $\cos^2 x = (1+ \cos 2x)/2$

$\cos^2 x - \sin^2 x = \cos 2x$ $\quad$ $\sin^2 x = (1- \cos 2x)/2$

$2 \sin x \cos x = \sin 2x$ $\quad$ $1 + \tan^2 x = \sec^2 x$

8th **Try integration by parts**

$u =$ $\quad$ $dv =$ ____dx

$du =$ ____dx $\quad$ $v =$ $\quad$ $\Rightarrow uv - \int v\, du$

9th **Refer to integral tables**

Find a match, adjust using *u*-substitution

Use an electronic, algebraic solver

10th **If integral is definite, use numerical methods**

Left- and Right-Hand Rectangles

Midpoint Rectangles

Trapezoidal Rule

Simpson's Rule