Infinite Series and Mathematica

The following lesson is designed to show you how to work with series in *Mathematica* and to improve your understanding of infinite series as a sequence of partial sums.
(*Don't forget*: To execute code in *Mathematica*, press **Shift-Ente**r or just **Enter on the number pad**.)

Visualizing Sequences

Let's start by exploring the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

- To see the first 20 terms in the sequence $\left\{\frac{1}{n^2}\right\}$ type and execute: `Table[1/n^2,{n,1,20}]`
- If you prefer decimals, type and execute **N[%]** for numerical approximations of the previous output.
- Create a list of coordinates to graph: `Table[{n,1/n^2},{n,1,20}]`
- To graph these points type and execute: `ListPlot[Table[{n,1/n^2},{n,1,20}]]`
  Note: You can copy-and-paste the line above into the new code after `ListPlot[`

- Repeat these steps for the harmonic series. To make this graph red, type `,PlotStyle->Hue[1]` just before the last `]` in the `ListPlot` command. *Don't forget the comma*. To make the points larger, insert `,PlotStyle->{Hue[1],PointSize[0.02]}` instead.
- What do you notice about each of these *sequences*? Which "test" does this relate to for the *series*? With the analysis thus far, what can we say about the convergence or divergence of the *series*?

Finding the Sums

- To add the first 20 terms in the *p*-series, type and execute: `Sum[1/n^2,{n,1,20}]`
- Move the cursor back to the `Sum` command, change the `20` to `200`, and re-execute.
- Try `2000`. Yikes! That's the largest fraction I've ever seen!
- To get a numerical approximation to this most recent output, type and execute: `N[%]`

- Let's create the sequence of the first 20 partial sums for the *p*-series.
  - Type and execute `Table[Sum[1/n^2,{n,1,m}],{m,1,20}]//N`
    (The `//N` converts the output to decimal form without seeing the fractions first.)
  - What do you notice about this new *sequence*? Does it appear to be converging? If so, to what?
- Change the `20` to `200`, and re-execute. Try `2000`. What do you think now?
- For a convergent series, *Mathematica* can find the infinite sum:
  `Sum[1/n^2,{n,1,Infinity}]`
  Find the numerical approximation for this value. How does it compare to S_{2000}?
- Return to the line of code that generated the first 2000 partial sums.
  Change `{m,1,2000}` to `{m,1,3000,100}` This counts to 3000 by 100s. Try to 5000 by 1000s.
  (Note: If you make the numbers too large, it will take a long time to calculate.)

- Repeat this section for the harmonic series.

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Graphing the Sequence of Partial Sums

- Return to the end of your code.
- To graph the sequence of partial sums, we will follow the same basic pattern. However we're going to name the graphs this time so we can use them later. First the *p*-series.
  `plot1 = ListPlot[Table[{m, Sum[1/n^2, {n, 1, m}]}, {m, 1, 20}]]`
  This graphs the *sequence of the first 20 partial sums*. What do you notice about this sequence?
- Graph the sequence of the first 20 partial sums for the harmonic series in red.
  `plot2=ListPlot[Table[{m,Sum[1/n,{n,1,m}]},{m,1,20}],PlotStyle->Hue[1]]`
  What do you notice about this sequence of partial sums?

- To compare both sequences, let's graph them together: **`Show[plot1, plot2]`**
  Does this result make sense?

- To explore this further, change the *p*-series to $5/n^2$ and re-execute **`ListPlot`** and **`Show`**.
- Now what do you see? What do you think will happen as we plot more partial sums?

- Download the example code from Ms. Brown's web site to make this next part easier.
  www.abbymath.com ~ Mathematica Files ~ Find "Infinite Series and *Mathematica*" and download the example code.

- Execute the code. You should see the same three plots you just created.
- To plot the first 200 partial sums, change **`nmax=20;`** to **`nmax=200;`** and re-execute.
  Does this tell you anything new?
- Try **`nmax=2000;`** but also change **`countby=10;`** to **`countby=100;`**
- Now, what do you see? Try **`nmax=3000;`**

- What do you think would happen if you changed $5/n^2$ to $5000/n^2$? Or $5{,}000{,}000/n^2$?
- What does this tell you about series convergence?

Further Examples and Exploration

Use what you learned about infinite series and *Mathematica* to determine and illustrate the convergence or divergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{1+3n}\right)^{n}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sqrt{n}}{5n+1}$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}}$

The codes for these expressions are listed below. If you define it as a function $a(n)$ as below, you can work with it more easily. Also, you can find any term you want by inserting it into the function.

1. **`a[n_] = (n/(3n + 1))^n`**
2. **`a[n_] = (-1)^(n - 1)Sqrt[n]/(5n + 1)`**
3. **`a[n_] = Log[n]/Sqrt[n]`** (For ln in *Mathematica*, use **Log[***expression***]**)