Geometric Transformations with Matrices

A Demonstration with *Mathematica*

Getting Started

First, open a few packages you may need by typing the following. (Note the reverse apostrophes ` found near the "Esc" key.)

```
<<LinearAlgebra`MatrixManipulation`
<<Graphics`Animation`
```

Note: There are no spaces.

Execute this code by pressing **Shift-Enter** or just **Enter** on the number pad. There is no output.

Two Dimensions: Set Up and Graphing

- Define the *column* vector $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ using either "Create Table/Matrix/Palette..." from the right-click menu or the "Basic Calculations Palette" found in the "File" menu. *Note: In the following directions, column vectors will be written as* $\mathbf{v} = [3, 4]^T$.
- To output a particular entry in a matrix, use *matrixname*`[[`*i*`,` *j*`]]`. For example, enter `v[[2,1]]`.
- To graph vectors, we're going to create the function "VectorPlot." This is a function of vector *and* color to make graphing multiple vectors easier. Type the following code. Think about what it means as you enter it.

```
VectorPlot[vect_,color_]:=
ParametricPlot[{vect[[1,1]]t,vect[[2,1]]t},{t,0,1},
PlotStyle->{Hue[color],Thickness[0.01]}]
```

  When you execute this, there will be no output.
- Try your function. Enter `VectorPlot[v,0]`. "0" indicates color. Use values between 0 and 1 for hue.
- Now go back to the function definition and type the following option just inside the last bracket `]`
  `,DisplayFunction->Identity` (Don't forget the comma and press **Shift-Enter** to re-define it.)
  Now the VectorPlot output is hidden.
- Enter the following:

```
Show[VectorPlot[2v,0],VectorPlot[v,0.6],DisplayFunction->$DisplayFunction]
```

Two Dimensions: Matrix Transformations

- Define matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. What does the matrix do?
- Define vector $\mathbf{u} = A\mathbf{v}$. *(Don't forget to use a period to multiply matrices correctly.)*
- Plot both together using the `Show` command. Make **v** red and **u** purple.
- Use *Mathematica* to find a matrix *A2* that reflects a vector across the *y*-axis followed by a contraction with factor $k = 3/4$. *(Don't forget to use a period to multiply matrices correctly.)*
- Apply this to vector **v** and graph the result along with the other two. Make this new vector light blue.
  Note: You may insert the new VectorPlot command into the code you already typed. Or use copy-and-paste.

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Two Dimensions: Rotations

- Define a function for rotation matrices by entering the following:

  $$\texttt{Rot[t\_]:=}\begin{pmatrix}\texttt{Cos[t]} & \texttt{-Sin[t]}\\ \texttt{Sin[t]} & \texttt{Cos[t]}\end{pmatrix}$$

- For example, type **`Rot[Pi/6]`**. What does this do?
- Create a matrix *A3* that that reflects a vector across the *y*-axis, followed by a contraction with factor $k = ¾$, followed by a rotation of $\pi/6$. Apply this to **v** and graph it along with the others. Make this vector green.
- Output *A3* as a matrix by typing **`A3//MatrixForm`** and pressing **Shift-Enter**.

Two Dimensions: Assignment

- Open a new Word document. Type your name, date, period, and title at the top.
- Highlight your matrix *A3* and choose "Copy As...Bitmap" from the "Edit" menu or the Right-Click menu. Go to Word and paste the matrix. Type "A3=" in front of it.
  Note: Regular copying may look normal on the screen, but it may not print correctly.
- Copy-and-paste your graph of four vectors into the Word document too. Graphs should copy normally.
- Return to *Mathematica* and complete the following:
  1. Find the standard matrix, *A4*, for the stated composition of linear operators on R^2:
     A reflection about the *x*-axis, followed by a rotation of $5\pi/8$, followed by a dilation with factor $k = 3/2$.
     Put a copy of *A4* into your Word document.
  2. Define a vector in the second quadrant and apply the above composite transformation.
  3. Graph your original vector and the result on the same coordinate axes and copy into Word.
  4. Repeat #1-3 with a vector in the third quadrant and the following composite transformation:
     A rotation of π, followed by a projection onto the *x*-axis, followed by a rotation of $-\pi/10$.
     Compare the graph of your answer with a couple other students. What do you notice? Why?
     Type your answer in your Word document. You may want to try two columns to fit it on one page.

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Three Dimensions: Set Up and Graphing

- Define a new vector **v** = $[5, 2, 3]^T$.
- To graph vectors in 3D, we are going to define VectorPlot3D, a function of vector and color.
```
VectorPlot3D[vect_,color_]:=ParametricPlot3D[
{vect[[1,1]]t,vect[[2,1]]t,vect[[3,1]]t,{Hue[color],Thickness[0.01]}},
{t,0,1},DisplayFunction->Identity]
```
- To help us visualize the three-dimensional vectors define the following:
```
ThreeAxes=ParametricPlot3D[{{t,0,0},{0,t,0},{0,0,t}},{t,-5,5}]
```
- To graph your vector use the following. The various options help see the graph better. Remember, you can copy-and-paste these after you type them in once.
```
graph=Show[ThreeAxes,VectorPlot3D[v,0],PlotRange->{{-5,5},{-5,5},{-5,5}},
AxesLabel->{x,y,z},AxesEdge->{{-1,-1},{-1,-1},{-1,-1}},Boxed->False,
ViewPoint->{3,0.5,1.5}]
```
Later you may need to adjust the plot range.
- To change the view point, *triple-click* on the word **`ViewPoint`** in the code. Choose "3D ViewPoint Selector" from the "Input" menu. Adjust the cube to the position you want and select paste. Re-execute.
- To animate, enter **`SpinShow[graph]`**. Double-click the middle blue bracket on the right to stack the images. Double-click on the stack to animate. Click anywhere to stop it. The speed controls are on the bottom left of the window.

Three Dimensions: Rotations and Assignment

- Define the three rotation operators. For example, **`Rot3Dx[t_]:=`** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos[t]} & -\text{Sin[t]} \\ 0 & \text{Sin[t]} & \text{Cos[t]} \end{pmatrix}$
- Create the standard matrix, B, that rotates a vector about the x-axis through an angle $\pi/4$ followed by a rotation about the z-axis through an angle $2\pi/3$.
- Apply this matrix to **v** and graph both on the same plot. Does your result agree with your prediction?
- Create a matrix *B2* that performs the two rotations in the opposite order.
- Apply this to **v** and graph it along with the other two. Be sure to use different colors to tell them apart.
- Copy this graph and the two matrices into a new page in Word. Don't forget to use "Copy As..."
- What do you notice? Why is this important? Type your answers in Word.
- Find the matrix, *B3*, that reflects a vector about the xy-plane, followed by a projection onto the xz-plane, followed by a rotation of $\pi/3$ about the y-axis. Create a vector **v** in any octant, apply *B3,* and graph both. Paste your results into Word. Try graphing the intermediate vectors to understand it better.